JOURNAL OF ALGEBRA 143, 130–143 (1991)

Normal F-Pure Surface Singularities

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INTRODUCTION

An inclusion of Noetherian local rings $R \to S$ is called *pure* if for any R-module M, the map $M \to S \otimes_R M$ is injective. If S is finitely generated as an R-module, then purity is equivalent to R being a direct summand of S as an R-module. A local ring R in characteristic p is said to be *F-pure* if the Frobenius map $F\colon R \to R$ is pure. This notion first appears in [HR]. The notion that a variety in characteristic p is Frobenius split has also been introduced (see [MR]).

Let X be a normal projective surface over an algebraically closed field k of characteristic $p>0$. Let $P \in X$ be a singular point, with $R = \mathcal{O}_{P,X}$. Let $f\colon Y \to X$ be the minimal resolution of the singularity at P. If E is the reduced exceptional divisor, then no irreducible component of E is an exceptional curve of the first kind; i.e., if $E = \bigcup_{i=1}^{n} E_i$ with each E_i irreducible, then $E_i \cong \mathbf{P}^1 \Rightarrow E_i^2 \leqslant -2$. The graph of E is defined as follows: it has a vertex v_i associated to each irreducible component E_i of E and an edge joining v_i to v_j for each point of intersection of E_i and E_j. Put $Z = Y \times_X \operatorname{Spec} R$, so that $f\colon Z \to \operatorname{Spec} R$ is the minimal resolution of singularities of Spec R. We shall prove the following results:

THEOREM 1.1. *With the notation above, suppose R is F-pure. Then we have*

(a) *The natural map (obtained from the formal function theorem)*

$$(R^1 f_* \mathcal{O}_Z)_P \cong H^1(Z, \mathcal{O}_Z) \to H^1(E, \mathcal{O}_E)$$

is an isomorphism.

(b) *E is a divisor with normal crossings, and one of the following holds:*

(i) *E is an irreducible smooth ordinary elliptic curve.*

(ii) *E is an irreducible rational curve which has only one singularity, which is nodal.*

0021-8693/91 $3.00

(iii) $E_i \cong \mathbf{P}^1$ *for all* i, *and the graph of* E *is an* n*-gon, i.e.*, $(E_i \cdot E_j) = 1$ *if* $i - j \equiv 1 \pmod n$, *and* $=0$, *otherwise.*

(iv) $E_i \cong \mathbf{P}^1$ *for all* i, *and the graph of* E *is a tree; equivalently*, $H^1(E, \mathcal{O}_E) = 0$, *so that by* (a), R *has a rational singularity.*

THEOREM 1.2. *Let* $R = \mathcal{O}_{P,X}$ *be a normal surface singularity in characteristic* p. *Suppose that the reduced exceptional divisor has only normal crossings and satisfies one of the conditions* (i), (ii), *and* (iii) *of* (b) *in Theorem* 1.1. *Then* R *is F-pure.*

It follows from Theorem 1.1 that if R is a Gorenstein normal surface singularity in characteristic p which is F-pure, then R satisfies one of the conditions (i), (ii), and (iii) of (b), or is a rational double point. Using the classification of rational double points in characteristic p (see [A]) we compute easily that any rational double point in characteristic $p > 5$ is F-pure. Hence we obtain a classification of normal F-pure surface singularities in characteristic $p > 5$. K. I. Watanabe has obtained analogous results in the Gorenstein case, using different techniques (see [W]).

PRELIMINARIES

We collect here some basic facts about Frobenius split schemes which are needed below. From now on, F will denote the absolute Frobenius morphism.

Recall that an $\mathbf{F}_p$-scheme X ($\mathbf{F}_p$ = finite field of cardinality p) is *Frobenius split* if the natural map $\mathcal{O}_X \to F_* \mathcal{O}_X$ is a split inclusion of $\mathcal{O}_X$-modules (see [MR]). If $X = \operatorname{Spec} R$, where R is a local ring, this is equivalent to R being F-pure. A splitting $F_* \mathcal{O}_X \to \mathcal{O}_X$ is called a *Frobenius splitting.*

LEMMA 1. *Let* X *be a reduced equidimensional Gorenstein* k*-scheme, where* k *is a perfect field of characteristic* p. *There is a natural isomorphism of* $\mathcal{O}_X$*-modules*

$$F_* \omega_X^{\otimes 1-p} \cong \operatorname{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X),$$

where ω_X *is the dualizing sheaf. In particular, Frobenius splittings of* X *are induced by elements of* $H^0(X, \omega_X^{\otimes 1-p})$.

Proof. This is a slight generalization of the results of [MR]. Duality for the Frobenius morphism yields an isomorphism (see [Ha, Chap. 3, Exercise 6.10])

$$F^! \omega_X \cong \omega_X,$$

giving isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) &\cong \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \omega_X)\otimes\omega_X^{-1}\\ &\cong F_*(F^!\omega_X)\otimes\omega_X^{-1}\cong F_*(\omega_X)\otimes\omega_X^{-1}\\ &\cong F_*(\omega_X\otimes F^*\omega_X^{-1})\cong F_*(\omega_X\otimes\omega_X^{\otimes -p})\cong F_*\omega_X^{\otimes 1-p}. \quad \blacksquare\end{aligned}$$

LEMMA 2. *Let X be a reduced equidimensional Gorenstein k-scheme for a perfect field k, and let U be an open dense subscheme. Suppose that U is Frobenius split, and the corresponding element of $H^0(U, \omega_U^{\otimes 1-p})$ extends to a global section of $\omega_X^{\otimes 1-p}$ on X. Then X is Frobenius split.*

Proof. The global section of $\omega_X^{\otimes 1-p}$ yields (by Lemma 1) an $\mathcal{O}_X$-linear map $F_*\mathcal{O}_X\to\mathcal{O}_X$, which restricts to a Frobenius splitting of U. The composite

$$\mathcal{O}_X\to F_*\mathcal{O}_X\to\mathcal{O}_X$$

equals multiplication by some $h\in H^0(X, \mathcal{O}_X)$; thus $h|_U=1$. Since U is dense, $h=1$, i.e., X is Frobenius split. $\blacksquare$

Next, we recall the notion of compatible Frobenius splitting. If X is Frobenius split, a closed subscheme $Y\subset X$ is *compatibly split* by a splitting $\psi: F_*\mathcal{O}_X\to\mathcal{O}_X$ if ψ induces (by restriction) an $\mathcal{O}_Y$-linear map $F_*\mathcal{O}_Y\to\mathcal{O}_Y$, which is a Frobenius splitting of Y. This is equivalent to requiring that if $\mathcal{I}_Y$ is the ideal sheaf of $Y\subset X$, then $\psi(F_*\mathcal{I}_Y)=\mathcal{I}_Y$. If $\mathcal{I}_Y^{[p]}$ is the ideal sheaf generated by pth-powers of sections of $\mathcal{I}_Y$, so that $\mathcal{I}_Y\cdot F_*\mathcal{O}_X=F_*\mathcal{I}_Y^{[p]}$, let $[\mathcal{I}_Y^{[p]}:\mathcal{I}_Y]=\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{I}_Y/\mathcal{I}_Y^{[p]})$. One knows (see [F], [MR]) that if $\sigma\in H^0(X, \omega_X^{\otimes 1-p})$, and if X/k is smooth, then the induced map $\psi: F_*\mathcal{O}_X\to\mathcal{O}_X$ satisfies $\psi(F_*\mathcal{I}_Y)\subset\mathcal{I}_Y$ if and only if σ is a global section of the subsheaf

$$[\mathcal{I}_Y^{[p]}:\mathcal{I}_Y]\otimes\omega_X^{\otimes 1-p}\subset\omega_X^{\otimes 1-p}$$

(in the case of local rings, this is a reformulation of Fedder's criterion). It is easy to see that if $\sigma\in H^0(X, [\mathcal{I}_Y^{[p]}:\mathcal{I}_Y]\otimes\omega_X^{\otimes 1-p})$, then the corresponding map $\psi: F_*\mathcal{O}_X\to\mathcal{O}_X$ satisfies $F_*(\mathcal{I}_Y)\subset\mathcal{I}_Y$, which is the only fact needed below. To see this, we make explicit the correspondence between σ and ψ (following [MR]). Let

$$C: F_*\omega_X\to\omega_X$$

be the trace map for Grothendieck duality for the Frobenius map; given the isomorphism $F^!\omega_X\cong\omega_X$, and the isomorphism $F_*F^!\omega_X\cong \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \omega_X)$, the map C is identified with the natural map

$$\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \omega_X)\to\omega_X$$

obtained by evaluation at 1. We use the letter C to denote the trace, since it coincides with the Cartier operator when X is smooth over k (see Section 2 of [MR], and the references given there). If ω is a local section generating ω_X, then we can locally write $\sigma = f\omega^{1-p}$, for some function f. Then ψ is locally given by

$$\psi(a) = b, \qquad \text{where} \quad C(af\omega) = b\omega.$$

Using the identity $C(a^p\eta) = aC(\eta)$ for any local sections a of $\mathcal{O}_X$ and η of ω_X, one can verify directly that ψ is independent of the choice of local generator ω. Now $\sigma \in H^0(X, [\mathcal{I}_Y^{[p]} : \mathcal{I}_Y] \otimes \omega_X^{\otimes 1-p})$ if and only if all the local expressions $f\omega^{1-p}$ for σ have the property that f is a local section of $[\mathcal{I}_Y^{[p]} : \mathcal{I}_Y]$. But if a is a local section of $\mathcal{I}_Y$, then af is a local section of $\mathcal{I}_Y^{[p]}$, and so $C(af\omega)$ is a local section of $\mathcal{I}_Y \otimes \omega_X$, i.e., b is a local section of $\mathcal{I}_Y$. Thus $\psi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$ as claimed.

In particular, consider the case when X/k is smooth and $Y \subset X$ is a reduced (effective) Cartier divisor. Then $\mathcal{I}_Y = \mathcal{O}_X(-Y)$, $\mathcal{I}_Y^{[p]} = \mathcal{O}_X(-pY)$, and $[\mathcal{I}_Y^{[p]} : \mathcal{I}_Y] = \mathcal{O}_X((1-p)Y)$. There is a natural map (adjunction formula)

$$\varphi : H^0(X, \omega_X(Y)^{\otimes 1-p}) \to H^0(Y, \omega_Y^{\otimes 1-p}).$$

LEMMA 3. *Let $s \in H^0(X, \mathcal{O}_X((1-p)Y) \otimes \omega_X^{\otimes 1-p}) = H^0(X, \omega_X(Y)^{\otimes 1-p})$, so that s induces a map $\bar{\psi} : F_*\mathcal{O}_Y \to \mathcal{O}_Y$. Then $\bar{\psi}$ is also the map $F_*\mathcal{O}_Y \to \mathcal{O}_Y$ induced by $\varphi(s) \in H^0(Y, \omega_Y^{\otimes 1-p})$.*

Proof. It suffices to prove that the two maps $F_*\mathcal{O}_Y \to \mathcal{O}_Y$ agree on a dense set of smooth points of Y. Hence we may assume without loss of generality that Y is smooth. Further, the equality of the two maps can be checked locally on Y; hence we may assume without loss of generality that if $n = \dim X = 1 + \dim Y$, then there exist n regular functions $x_1, ..., x_n$ on X, such that Y is defined by $x_n = 0$, and $x_1, ..., x_n$ yield an étale map $X \to \mathbf{A}_k^n$ (i.e., $\Omega_{X/k}^1$ is trivial with basis $dx_1, ..., dx_n$). We identify C with the Cartier operator (see [C] for the definition and properties of the Cartier operator; the relation with splittings is explained in [MR]), which can be made explicit as follows. The functions $x_1, ..., x_n$ form a p-basis for $\mathcal{O}_X$, in the sense that $F_*\mathcal{O}_X$ is a free $\mathcal{O}_X$-module with a basis given by the monomials

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \qquad 0 \leqslant i_j \leqslant p-1.$$

Then

$$C(f\, dx_1 \wedge \cdots \wedge dx_n) = h\, dx_1 \wedge \cdots \wedge x_n,$$

where

$$h^p = \text{coefficient of } (x_1 x_2 \cdots x_n)^{p-1} \text{ in the expression for } f \text{ in terms of the } p\text{-basis.}$$

Now suppose $s = f(dx_1 \wedge \cdots \wedge dx_n)^{1-p}$. Then ψ has the explicit description

$$\psi(a) = (\text{coefficient of } (x_1 x_2 \cdots x_n)^{p-1}) \text{ in } af)^{1/p}.$$

Since $s \in H^0(X, \omega_X(Y)^{\otimes 1-p})$, we have $f = g x_n^{p-1}$. Thus

$$\begin{aligned}\psi(a) &= (\text{coefficient of } (x_1 x_2 \cdots x_n)^{p-1} \text{ in } a g x_n^{p-1})^{1/p} \\ &= (\text{coefficient of } (x_1 x_2 \cdots x_{n-1})^{p-1} \text{ in } ag)^{1/p}.\end{aligned}$$

Thus, if an overbar denotes restriction to Y,

$$\bar{\psi}(\bar{a}) = (\text{coefficient of } (\bar{x}_1 \bar{x}_2 \cdots \bar{x}_{n-1})^{p-1} \text{ in } \overline{ag})^{1/p},$$

where we note that $\bar{x}_1, ..., \bar{x}_{n-1}$ give a p-basis for $\mathcal{O}_Y$.

The map φ: $H^0(X, \omega_X(Y)^{\otimes 1-p}) \to H^0(Y, \omega_Y^{\otimes 1-p})$ satisfies

$$\varphi(g x_n^{p-1}(dx_1 \wedge \cdots \wedge dx_n)^{1-p}) = \bar{g}(d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_{n-1})^{1-p},$$

since under the adjunction map, $dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dx_n/x_n \mapsto d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_{n-1}$. The map $F_* \mathcal{O}_Y \to \mathcal{O}_Y$ induced by $\varphi(s)$ is given by the Cartier operator on Y, which we compute using the p-basis $\bar{x}_1, ..., \bar{x}_{n-1}$ as

$$C(\bar{f}\, d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_{n-1}) = \bar{h}\, d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_{n-1},$$

where

$$\bar{h}^p = \text{coefficient of } (\bar{x}_1 \bar{x}_2 \cdots \bar{x}_{n-1})^{p-1} \text{ in } \bar{f}.$$

Thus $\varphi(s)$ induces the map given by the formula

$$a \mapsto (\text{coefficient of } (\bar{x}_1 \bar{x}_2 \cdots \bar{x}_{n-1})^{p-1} \text{ in } \overline{ag}),$$

which is precisely the formula for $\bar{\psi}$. ∎

1. Proof of Theorem 1.1

We first prove that Z is Frobenius split. If $U = Z - E$, then U is isomorphic to the punctured spectrum of R. Since R is F-pure, Spec R is

Frobenius split, and hence U is a Frobenius split, dense open subscheme of Z. Hence, by Lemma 2, it suffices to prove that

$$H^0(U, \omega_U^{1-p}) \cong H^0(Z, \omega_Z^{1-p}).$$

Let $M = H^0(U, \omega_U^{1-p})$, and let $\tilde{M}$ be the associated coherent sheaf on Spec R. If $g\colon Z \to \operatorname{Spec} R$ is induced by $f\colon Y \to X$, then $g^*\tilde{M}$ is a coherent $\mathcal{O}_Z$-module generated by its global sections and satisfies $g^*\tilde{M}|_U \cong \omega_U^{\otimes 1-p}$. Let $\mathcal{L} = ((g^*\tilde{M})^\vee)^\vee$ (where the superscript $^\vee$ denotes the functor $\mathcal{H}om_{\mathcal{O}_Z}(-, \mathcal{O}_Z)$). The natural map $g^*\tilde{M} \to \mathcal{L}$ has a cokernel supported on a finite set of closed points of E, namely the points where $g^*\tilde{M}/(\text{torsion})$ is not locally free. Also, $\mathcal{L}|_U \cong \omega_U^{\otimes 1-p}$, and $\mathcal{L} \in \operatorname{Pic} Z$. From the exact sequence

$$0 \to \bigoplus_{i=1}^{n} \mathbf{Z}[\mathcal{O}_Z(E_i)] \to \operatorname{Pic} Z \to \operatorname{Pic} U \to 0$$

we see that there is an isomorphism

$$\mathcal{L} \cong \omega_Z^{\otimes 1-p}\left(\sum_{i=1}^{n} a_i E_i\right)$$

for unique integers a_i (the classes $[\mathcal{O}_Z(E_i)] \in \operatorname{Pic} Z$ are $\mathbf{Z}$-linearly independent since the intersection matrix $((E_i \cdot E_j))_{1 \leqslant i,j \leqslant n}$ is negative definite—see [Mu]). Further, the natural sheaf map

$$H^0(Z, \mathcal{L}) \otimes_R \mathcal{O}_Z \to \mathcal{L}$$

has cokernel supported at a finite set of closed points. In particular, the map

$$H^0(E_i, \mathcal{L}|_{E_i}) \otimes_k \mathcal{O}_{E_i} \to \mathcal{L}|_{E_i}$$

has cokernel supported at a finite set of closed points, for each exceptional component E_i. Hence $\deg \mathcal{L}|_{E_i} \geqslant 0$, with equality if and only if $\mathcal{L}|_{E_i} \cong \mathcal{O}_{E_i}$.

Now $\omega_Z|_{E_i} \cong \omega_{E_i}(-E_i)$ by the adjunction formula, and $\deg \omega_{E_i} = 2p_a(E_i) - 2$, where $p_a(E_i) = \dim H^1(E_i, \mathcal{O}_{E_i})$ is the arithmetic genus of E_i. Hence

$$\deg \mathcal{L}|_{E_i} = (1-p)(2p_a(E_i) - 2 - E_i^2) + \sum_{j=1}^{n} a_j (E_i \cdot E_j)$$

(where $\deg \mathcal{O}_{E_i}(-E_i) = -(E_i \cdot E_i) = -E_i^2$). Thus

$$\left(\left(\sum_{j=1}^{n} a_j E_j\right) \cdot E_i\right) \geqslant (p-1)(2p_a(E_i) - 2 - E_i^2).$$

Now $p_a(E_i) \geqslant 0$ with equality precisely when $E_i \cong \mathbf{P}^1$, in which case $E_i^2 \leqslant -2$. Hence in any case $2p_a(E_i) - 2 - E_i^2 \geqslant 0$ with equality $\Rightarrow E_i \cong \mathbf{P}^1$ and $E_i^2 = -2$.

Let

$$C = \sum_{a_i > 0} a_i E_i, \qquad D = \sum_{a_i \leqslant 0} -a_i E_i$$

so that C, D are effective divisors without common components, and

$$\sum_{i=1}^{n} a_i E_i = C - D.$$

Thus, we have inequalities for all i,

$$(C - D \cdot E_i) \geqslant 0.$$

Hence

$$(C - D \cdot C) \geqslant 0.$$

But

$$(C - D \cdot C) = C^2 - (D \cdot C) \leqslant C^2$$

(since C, D have no common component, $(C \cdot D) \geqslant 0$). Hence $C^2 \geqslant 0$. Since the intersection matrix $((E_i \cdot E_j))_{1 \leqslant i,j \leqslant n}$ is negative definite (see [Mu]), we must have $C = 0$, and so $\mathscr{L} \cong \omega_Z^{\otimes 1-p}(-D)$, where D is effective. From the definition of $\mathscr{L}$ we thus have equalities

$$H^0(U, \omega_U^{\otimes 1-p}) = H^0(Z, \mathscr{L}) = H^0(Z, \omega_Z^{\otimes 1-p}) = H^0(Z, \omega_Z^{\otimes 1-p}(-D)).$$

By Lemma 2, this implies in particular that Z is Frobenius split.

We now prove 1.1(a). From the formal function theorem, we have an isomorphism (see [Ha])

$$(R^1 f_* \mathcal{O}_Z)_P \cong H^1(Z, \mathcal{O}_Z) \cong \varprojlim_m H^1(mE, \mathcal{O}_{mE}),$$

where mE is the subscheme with ideal sheaf $\mathcal{O}_Z(-mE)$. Further, the maps in the inverse system are all surjective and are isomorphisms for large m. Fix an integer m such that $H^1(Z, \mathcal{O}_Z) \cong H^1(mE, \mathcal{O}_{mE})$, and choose N with $p^N \geqslant m$. If F^N is the N-fold iterated Frobenius map, we have a factorization

$$\begin{tikzcd} \mathcal{O}_{mE} \arrow[r, "F^N"] \arrow[d] & \mathcal{O}_{mE} \\ \mathcal{O}_E \arrow[ur, "h"'] & \end{tikzcd}$$

where the vertical arrow is induced by restriction. Thus there is a diagram

$$\begin{tikzcd}
H^1(Z, \mathcal{O}_Z) \arrow[r, "\cong"] \arrow[d, "F_*^N"] & H^1(mE, \mathcal{O}_{mE}) \arrow[r, two heads] \arrow[d, "F_*^N"] & H^1(E, \mathcal{O}_E) \arrow[dl, "h_*"] \arrow[d, "F_*^N"] \\
H^1(Z, \mathcal{O}_Z) \arrow[r, "\cong"] & H^1(mE, \mathcal{O}_{mE}) \arrow[r, two heads] & H^1(E, \mathcal{O}_E)
\end{tikzcd}$$

Since Z is Frobenius split, F_*^N is injective on $H^1(Z, \mathcal{O}_Z)$. Hence $H^1(Z, \mathcal{O}_Z) \twoheadrightarrow H^1(E, \mathcal{O}_E)$ must be injective, proving (a). We also see that the Frobenius action on $H^1(E, \mathcal{O}_E)$ is injective.

To prove 1.1(b) we further analyze the inequalities obtained earlier. Since $a_j \leqslant 0$ for all j, we have for any i

$$a_i E_i^2 \geqslant \sum_{j=1}^{n} a_j (E_j \cdot E_i) \geqslant (p-1)(2p_a(E_i) - 2 - E_i^2), \qquad (*)_i$$

where the right side is non-negative and vanishes if and only if $E_i \cong \mathbf{P}^1$, $E_i^2 = -2$. Hence if $a_i = 0$ for some i, then all the inequalities in $(*)_i$ must be equalities, $E_i \cong \mathbf{P}^1$ and $E_i^2 = -2$. Thus $a_j = 0$ for all j such that $(E_i \cdot E_j) \neq 0$, $j \neq i$. Since E is connected (see [Mu]), we deduce that either $a_i = 0$ for *all* i, or $a_i < 0$ for *all* i.

If $a_i = 0$ for all i, then $E_i \cong \mathbf{P}^1$, and $E_i^2 = -2$, for all i. Then we claim that E has normal crossings, and the graph of E is a tree. Indeed, if either of these statements is false, we can find a non-zero reduced effective divisor C, which is a linear combination of the E_i, such that $(C - E_i \cdot E_i) \geqslant 2$ for all $E_i \subset C$, and so $C^2 \geqslant 0$, contradicting negative definiteness. Thus, $E_i \cong \mathbf{P}^1$, $E_i^2 = -2$ for all i, and $H^1(E, \mathcal{O}_E) = H^1(Z, \mathcal{O}_Z) = 0$, i.e., R has a rational double point singularity (see [D]). This is a particular case of 1.1(b)(iv).

Next, consider the case when $a_j < 0$ for all j. Suppose that $p_a(E_i) \geqslant 1$ for some i, so that

$$(-a_i)(-E_i^2) \geqslant (p-1)(2p_a(E_i) - 2 - E_i^2) \geqslant (p-1)(-E_i^2);$$

i.e.,

$$-a_i \geqslant p-1 \qquad (\text{as } E_i^2 < 0).$$

Thus, any global section of $\omega_Z^{\otimes 1-p}$ must vanish to order at least $p-1$ along E_i. Hence E_i is compatibly Frobenius split in Z. If $\sigma \in H^0(Z, \omega_Z(E_i)^{\otimes 1-p})$ gives a Frobenius splitting of Z, then by Lemma 3, the splitting of E_i is induced by the image of σ under the natural map

$$\psi\colon H^0(Z, \omega_Z(E_i)^{\otimes 1-p}) \to H^0(E_i, \omega_{E_i}^{\otimes 1-p}).$$

In particular $\psi(\sigma) \in H^0(E_i, \omega_{E_i}^{\otimes 1-p})$ is a non-zero section. Since in fact σ

vanishes to order at least $-a_j$ along E_j, $\psi(\sigma)$ vanishes at all points of $(E-E_i)\cap E_i$. Now $\deg \omega_{E_i}^{\otimes 1-p}=(1-p)\deg \omega_{E_i}\geqslant 0$ as $H^0(E_i, \omega_{E_i}^{\otimes 1-p})$ has a non-zero section. Hence $\deg \omega_{E_i}=2p_a(E_i)-2\leqslant 0$, i.e., $\deg \omega_{E_i}=0$, and $p_a(E_i)=1$, since we assumed $p_a(E_i)\geqslant 1$. Hence E_i is either a smooth elliptic curve, or an irreducible rational curve with only one singular point, which is either a node or an ordinary cusp; further $\omega_{E_i}\cong \mathcal{O}_{E_i}$. Thus $\omega_{E_i}^{\otimes 1-p}\cong \mathcal{O}_{E_i}$, so that a non-zero global section has no zeroes. Hence $(E-E_i)\cap E_i$ is empty, i.e., $E=E_i$, since E is connected (see [D]). Since E is Frobenius split, the Frobenius action on $H^1(E, \mathcal{O}_E)$ is injective; hence E cannot have a cusp, and if E is an elliptic curve, then it is ordinary. Thus E is either an ordinary elliptic curve or an irreducible rational curve with one node, as in 1.1(b)(i), (ii).

We are now left with the case when $a_j<0$, and $E_j\cong \mathbf{P}^1$, for all j. We need the following lemma.

LEMMA 4. *Let $C=C_1+\cdots+C_m$ be a reduced effective divisor on a smooth surface, with dualizing sheaf ω_C, and irreducible components C_i. Suppose that for each i, $C-C_i$ is non-empty and connected and satisfies $(C-C_i\cdot C_i)\geqslant 2$. Then for each i,*

$$\deg \omega_C|_{C_i}\geqslant 0,$$

with equality only if $\omega_C|_{C_i}\cong \mathcal{O}_{C_i}$.

Proof. It suffices to show that for any "general" point $P\in C$ (which we may take to be a smooth point, in particular)

$$\dim H^0(C, \omega_C(-P))=\dim H^0(C, \omega_C)-1.$$

By Serre duality on C, this is equivalent to

$$\dim H^1(C, \mathcal{O}_C(P))=\dim H^1(C, \mathcal{O}_C)-1.$$

From the exact sequence

$$0\to \mathcal{O}_C\to \mathcal{O}_C(P)\to i_{P*}k(P)\to 0$$

(where $i_{P*}k(P)$ is the skyscraper sheaf at P whose stalk is the residue field $k(P)$), we are reduced to showing that

$$H^0(C, \mathcal{O}_C)=H^0(C, \mathcal{O}_C(P))=k,$$

i.e., that a global meromorphic function on C, which is regular on $C-\{P\}$, and has at worst a simple pole at P, is a constant.

Let $P\in C_i$; since P is a smooth point, $P\notin C_j$ for $j\neq i$. If h is a meromorphic function as described above, $h|_{C-C_i}$ is constant since it is a regular function on a connected, reduced projective variety. Subtracting a constant from h, we may assume that h vanishes on $C-C_i$. Thus $h|_{C_i}$ is a meromorphic function, regular on $C_i-\{P\}$, which vanishes on the subscheme $(C-C_i)\cap C_i$ of C_i and has at worst a simple pole at P. Now $(C-C_i\cap C_i)$ has a structure sheaf of length $(C-C_i\cdot C_i)\geqslant 2$. Hence either $h|_{C_i}=0$, or the divisor of zeroes of $h|_{C_i}$ has degree $\geqslant 2$. But the divisor of poles of $h|_{C_i}$ is of degree at most 1. ∎

We return to our situation, where we have to consider the case when $a_j<0$ and $E_j\cong\mathbf{P}^1$, for all j. Either E has only normal crossings and the graph of E is a tree (this is the case 1.1(b)(iv)), or else we can find a divisor C, which is a reduced linear combination of the E_i, and satisfies the hypothesis of Lemma 4. We can write

$$\omega_Z^{\otimes 1-p}\left(\sum_{i=1}^n a_iE_i\right)=\omega_Z(C)^{\otimes 1-p}\left(\sum_{i=1}^n a_iE_i+(p-1)C\right),$$

where

$$\sum_{i=1}^n a_iE_i+(p-1)C=\sum_{E_i\not\subset C} a_iE_i+\sum_{E_i\subset C}(a_i+p-1)E_i.$$

Now $-a_i\leqslant p-1$ for all i, since for any effective divisor D, a section of $\omega_Z^{\otimes 1-p}(-pD)$ gives a map $F_*\mathcal{O}_Z\to\mathcal{O}_Z$ which factors through the ideal sheaf $\mathcal{O}_Z(-D)$ and hence cannot give a Frobenius splitting of Z. Thus $a_i+p-1\geqslant 0$ for all i. Now $\omega_Z(C)|_{C_i}=\omega_C|_{C_i}$ for $E_i\subset C$. Hence by Lemma 4, $\deg\omega_Z(C)|_{E_i}\geqslant 0$ for $E_i\subset C$. Thus, for $E_i\subset C$, we have the inequality

$$\begin{aligned}\deg\omega_Z^{\otimes 1-p}\left(\sum a_jE_j\right)&=(1-p)\deg\omega_C|_{E_i}+\sum_{E_j\not\subset C}a_j(E_j\cdot E_i)\\&\quad+\sum_{E_j\subset C}(p-1+a_j)(E_j\cdot E_i)\geqslant 0,\end{aligned}$$

where the first two terms of the middle expression are $\leqslant 0$. Hence

$$\sum_{E_j\subset C}(p-1+a_j)(E_j\cdot E_i)\geqslant 0\qquad\text{for all}\quad E_i\subset C.$$

Thus

$$D=\sum_{E_j\subset C}(p-1+a_j)E_j$$

satisfies $D^2\geqslant 0$, i.e., $D=0$, by negative definiteness. Hence $-a_j=p-1$ for all $E_j\subset C$, so that C is compatibly Frobenius split in Z. Thus

$H^0(C, \omega_C^{\otimes 1-p})$ has a non-zero section, which does not vanish identically on any component. Since $\deg \omega_C|_{E_i} \geqslant 0$ for all $E_i \subset C$, by Lemma 4, we must have $\omega_C \cong \mathcal{O}_C$.

If

$$D' = \sum_{E_j \not\subset C} -a_j E_j,$$

then

$$\omega_Z^{\otimes 1-p}\left(\sum a_i E_i\right) = \omega_Z(C)^{\otimes 1-p}(-D').$$

Since

$$H^0\left(Z, \omega_Z^{\otimes 1-p}\left(\sum a_i E_i\right)\right) = H^0(Z, \omega_Z(C)^{\otimes 1-p}(-D'))$$
$$= H^0(Z, \omega_Z^{\otimes 1-p}) = H^0(Z, \omega_Z(C)^{\otimes 1-p}),$$

the image of any section of $\omega_Z(C)^{\otimes 1-p}$ under

$$H^0(Z, \omega_Z(C)^{\otimes 1-p}) \to H^0(C, \omega_C^{\otimes 1-p}) = H^0(C, \mathcal{O}_C)$$

must vanish on the subscheme $D' \cap C$. Since this restriction map is not 0 (by Lemma 3), $D' \cap C$ must be empty. Since $a_j < 0$ for all j, and E is connected, this forces $C = E$.

For any i, $E - E_i$ and E_i are closed subschemes of Z which are compatibly Frobenius split by any given Frobenius splitting of Z. In particular, their scheme theoretic intersection $(E - E_i) \cap E_i$ is compatibly Frobenius split in Z and is hence reduced. This implies that E has only normal crossings. Finally, since $C = E$, $\omega_E \cong \mathcal{O}_E$, and so

$$\omega_Z|_{E_i} \cong \omega_Z(E)|_{E_i} \otimes \mathcal{O}_{E_i}(-E) \cong \omega_E|_{E_i} \otimes \mathcal{O}_{E_i}(-E) \cong \mathcal{O}_{E_i}(-E).$$

Hence $\deg \omega_Z|_{E_i} = -(E_i \cdot E)$. But

$$\omega_Z|_{E_i} \cong \omega_Z(E_i)|_{E_i} \otimes \mathcal{O}_{E_i}(-E_i) \cong \omega_{E_i}(-E_i).$$

Hence $\deg \omega_Z|_{E_i} = -2 - E_i^2$, since $E_i \cong \mathbf{P}^1$, and so $(E - E_i \cdot E_i) = 2$ for all i. One sees easily that the graph of E must then be an n-gon, i.e., E is as in 1.1(b)(iii). This finishes the proof of Theorem 1.1. ∎

2. Proof of Theorem 1.2

Let $\hat{R}$ be the completion of R, and let $\hat{Z} = Z \times_{\operatorname{Spec} R} \operatorname{Spec} \hat{R}$, $\hat{\pi}: \hat{Z} \to Z$, and let $\hat{E}$ be the reduced exceptional divisor of $\hat{\pi}$. Then under the morphism $p_1: \hat{Z} \to Z$, we have an isomorphism $\hat{E} \to E$. Further, the map of ideal class groups $\mathrm{Cl}(R) \to \mathrm{Cl}(\hat{R})$ is injective. Last, $p_1^*: H^1(Z, \mathcal{O}_Z) \to H^1(\hat{Z}, \mathcal{O}_{\hat{Z}})$ is an isomorphism, since $H^1(Z, \mathcal{O}_Z)$ is an Artinian R-module, and $H^i(\hat{Z}, \mathcal{O}_{\hat{Z}}) = H^i(Z, \mathcal{O}_Z) \otimes_R \hat{R}$ from the formal function theorem.

Let $n\hat{E}$ be the effective divisor in $\hat{Z}$ with ideal sheaf $\mathcal{O}_{\hat{Z}}(-n\hat{E})$. The Grothendieck existence theorem (see [G]) gives an isomorphism

$$\mathrm{Pic}\, \hat{Z} \cong \varprojlim_n \mathrm{Pic}(n\hat{E}).$$

Now if the singularity R satisfies one of the conditions 1.1(b)(i), (ii), and (iii), then $\omega_{\hat{E}} \cong \mathcal{O}_{\hat{E}}$, and for $n > 0$, $\mathcal{O}_{\hat{Z}}(-n\hat{E}) \otimes \mathcal{O}_{\hat{E}}$ is a line bundle on $\hat{E}$ which has strictly positive degree, and non-negative degree when restricted to each irreducible component. Hence the dual line bundle $\mathcal{O}_{\hat{E}}(n\hat{E})$ has no non-zero global sections, so that by Serre duality, $H^1(\hat{E}, \mathcal{O}_{\hat{E}}(-n\hat{E})) = 0$. From the exact sequences for each $n > 0$

$$0 \to \mathcal{O}_{\hat{E}}(-n\hat{E}) \to \mathcal{O}_{(n+1)\hat{E}} \to \mathcal{O}_{n\hat{E}} \to 0$$

$$0 \to \mathcal{O}_{\hat{E}}(-n\hat{E}) \to \mathcal{O}^*_{(n+1)\hat{E}} \to \mathcal{O}^*_{n\hat{E}} \to 0$$

we thus obtain isomorphisms

$$\mathrm{Pic}\, \hat{Z} \cong \varprojlim_n \mathrm{Pic}(n\hat{E}) \cong \mathrm{Pic}\, \hat{E}.$$

In particular, the restriction map $\mathrm{Pic}\, Z \to \mathrm{Pic}\, E$ is injective. Since $\omega_Z(E) \otimes \mathcal{O}_E \cong \mathcal{O}_E$ from the adjunction formula (and $\omega_E \cong \mathcal{O}_E$), we see that $\omega_Z(E) \cong \mathcal{O}_Z$. Thus the exact sequence

$$0 \to \omega_Z(E)^{\otimes 1-p} \otimes \mathcal{O}_Z(-E) \to \omega_Z(E)^{\otimes 1-p} \to \omega_E^{\otimes 1-p} \to 0$$

is isomorphic to

$$0 \to \mathcal{O}_Z(-E) \to \mathcal{O}_Z \to \mathcal{O}_E \to 0.$$

In particular, the natural map

$$H^0(Z, \omega_Z(E)^{\otimes 1-p}) \to H^0(E, \omega_E^{\otimes 1-p})$$

is surjective, and the generator of $H^0(E, \omega_E) \cong k$ lifts to a global section of $\omega_Z(E)^{\otimes 1-p}$, which we may regard as a global section of $\omega_Z^{\otimes 1-p}$ that vanishes to order at least $p-1$ along E. By direct computation, the map $F_*\mathcal{O}_E \to \mathcal{O}_E$ induced by a generator of $H^0(E, \omega_E^{\otimes 1-p})$ has the property that the induced composite map $\mathcal{O}_E \to \mathcal{O}_E$ is multiplication by a non-zero scalar (for example, it suffices to remark that E is Frobenius split). Thus we can find a section $\sigma \in H^0(Z, \omega_Z^{\otimes 1-p})$, vanishing to order $\geqslant p-1$ along E, so that if $\psi: F_* \mathcal{O}_Z \to \mathcal{O}_Z$ is the corresponding map, the composite $\mathcal{O}_Z \to \mathcal{O}_Z$ is multiplication by some $h \in H^0(Z, \mathcal{O}_Z) = R$, such that $h|_E$ is a non-zero scalar. Thus $h \in R^*$, and Z is Frobenius split. Hence

$$H^0(Z, \mathcal{O}_Z) \to H^0(Z, F_* \mathcal{O}_Z)$$

is split, i.e., R is F-pure.

Remark 1. Let E be an ordinary elliptic curve over an algebraically closed field of characteristic p, and let $t \in H^1(E, \mathcal{O}_E)$. Then t corresponds to an extension

$$0 \to \mathcal{O}_E \to \mathcal{V}_t \to \mathcal{O}_E \to 0.$$

Then an easy computation shows that the associated projective bundle $\mathbf{P}_E(\mathcal{V}_t)$ is *not* Frobenius split, if $t \neq 0$, while $\mathbf{P}_E(\mathcal{V}_0)$ *is* Frobenius split. Hence we obtain a one-parameter family of projective bundles $\{\mathbf{P}_E(\mathcal{V}_t)\}_{t \in H}$ parametrized by the vector space $H = H^1(E, \mathcal{O}_E)$, such that $\mathbf{P}_E(\mathcal{V}_t)$ is Frobenius split precisely for one parameter value $t = 0$. Taking the family of cones over $\mathbf{P}_E(\mathcal{V}_t)$ (say, associated to the projective embeddings given by the line bundles $\mathcal{O}_{\mathbf{P}_E(\mathcal{V}_t)}(n) \otimes \pi_t^* \mathcal{L}$, where $\mathcal{L}$ has degree $\geqslant 3$ on E, n is sufficiently large and $\pi_t: \mathbf{P}_E(\mathcal{V}_t) \to E$ is the projection), we obtain a family of normal local domains R_t of dimension 3, parametrized by $t \in H$, such that R_0 is F-pure, but R_t is *not* F-pure for $t \neq 0$. Our results suggest that this cannot happen in dimension 2.

Remark 2. The results of this paper are used in [S] to classify normal surface singularities of dimension 2 in characteristic 0 which are of F-pure type, in the sense of [H].

References

[A] M. Artin, Coverings of the rational double points in characteristic p, *in* "Complex Analysis and Algebraic Geometry" (W. L. Baily and T. Shioda, Eds.), pp. 11–22, Cambridge Univ. Press, Cambridge, 1977.

[C] P. Cartier, Questions de rationalité des diviseurs en géométrie algébrique, *Bull. Soc. Math. France* **86** (1958), 177–251.

[D] A. Durfee, Fifteen characterizations of rational double points, *Enseign. Math.* **25** (1979), 131–163.

[F] R. Fedder, F-purity and rational singularity, *Trans. Amer. Math. Soc.* **278** (1983), 461–480.

[G] A. Grothendieck, "SGA 2, Cohomologie locale des faisceaux coherents et théorèmes de Lefschetz locaux et globaux," North-Holland, Amsterdam, 1968.

[H] M. Hochster, F-pure algebras and algebras of F-pure type, *in* "Proceedings, Conf. on Comm. Algebra, Katata, Japan, 1981."

[HR] M. Hochster and J. Roberts, The purity of Frobenius and local cohomology, *Adv. in Math.* **21** (1976), 117–172.
[Ha] R. Hartshorne, "Algebraic Geometry," Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York, 1977.
[MR] V. B. Mehta and A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties, *Ann. of Math.* **122** (1985), 27–40.
[Mu] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, *Publ. Math. I.H.E.S.* **9** (1961), 5–22.
[S] V. Srinivas, Normal surface singularities of F-pure type, *J. Algebra* **142** (1991).
[W] K.-I. Watanabe, Study of F-purity in dimension two, *in* "Proceedings, Conf. on Algebraic Geometry in Honor of M. Nagata, Kinokuniya, Japan, pp. 791–800, 1988.